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# Some Remarks on the Use of Smoothing and Linear Diffusion Terms to Control Noise

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This is an unreviewed manuscript, primarily intended for informal exchange of information among NMC staff members.

Consider

$$\frac{\partial v}{\partial t} + U - K \nabla^2 v = 0 \tag{1}$$

where v is any variable whose noise is to be controlled by the last term on the left, and U represents all the terms in the equation not otherwise shown. The centered difference analogue of (1) is

$$\frac{v^{\tau+1} - v^{\tau-1}}{2\Delta t} + U^{\tau} - K \frac{\nabla^2 v^{\tau-1}}{\Delta^2} = 0$$

where $\tau$ is the serial number of the time step, $(\nabla^2 v)/\Delta^2$ is an as yet unspecified finite-difference estimate of the Laplacian, $\Delta$ is the distance between grid points, and $\Delta t$ is the period between time steps. For convenience, I re-write the last equation.

$$v^{\tau+1} = - 2 U^{\tau} \Delta t + v^{\tau-1} + \frac{2 K \Delta t}{\Delta^2} \nabla^2 v^{\tau-1} \tag{2}$$

The last two terms combined may be regarded as a smoothed value of v at $\tau$-1.

For example, in one dimension, if the simplest estimate of the Laplacian is used,

$$\nabla^2 v = \Delta^2 v_{xx}$$

Representing the smoothed v with a superposed bar, and letting j be the serial number of grid points in space, I write

$$\begin{aligned} \bar{v} &= v + 2 K \Delta t \, v_{xx} \\ &= v_j + \frac{2 K \Delta t}{\Delta^2} (v_{j+1} - 2 v_j + v_{j+1}) \end{aligned}$$

which is plainly a simple three-point smoothing operator with a weight at the central point of

$$\mu = 1 - \frac{4 K \Delta t}{\Delta^2}$$

Now if, as we have usually done at NMC, the horizontal diffusion term is not used to represent physical diffusion, but rather to control small-scale noise, what should be done when the resolution of a model is changed?

I suggest being guided by the working hypothesis that the parts of the spectrum in which noise will need to be controlled will be the same, as measured by wave-number, not wave-length. This leads to retaining the value of $\mu$, not K, as the resolution is changed. This principle also has the virtue of assuring all the benefits of higher resolution because the interesting longer wave lengths will be less affected by the control device with the higher resolution.

In two horizontal dimensions, what form should be used for the Laplacian? Consider first the form

$$\nabla^2 v \cong v_{xx} + v_{yy} \tag{3}$$

Then the last two terms of (2) are

$$\overline{v} = v + 2 K \Delta t (v_{xx} + v_{yy})$$

and the weight of the central point in this smoothing operation is

$$\mu = 1 - \frac{8 K \Delta t}{\Delta^2}$$

Applying this to

$$v = \exp i (r x + s y) \tag{4}$$

I find that

$$\overline{v} = v \{1 - (1 - \mu) [1 - \tfrac{1}{2} ( \cos r \Delta + \cos s \Delta)]\}$$

Thus, the form (3) is unsatisfactory because suppression of a high wave number in one dimension is subject to its wave number in the other dimension.

A satisfactory alternative is a 9-point finite-difference Laplacian derived from smoothing theory. Consider a 3-point smoothing operator in one dimension whose central weight is $\mu$. If this were applied indepently in each dimension, a component such as (4) with a high wave number in one dimension would be suppressed, regardless of its wave number in the other dimension. In fact, the result of both operations would be

$$\overline{v} = v [1 - (1 - \mu)(1 - \cos r \Delta)][1 - (1 - \mu)(1 - \cos s \Delta)]$$

The operations in the two dimensions can be combined into a single two-dimensional operation. The plan of the two dimensional smoothing operator is

$$\begin{array}{ccc} \frac{1}{4}(1-\mu)^2 & \frac{1}{2}\mu(1-\mu) & \frac{1}{4}(1-\mu)^2 \\ \\ \frac{1}{2}\mu(1-\mu) & \mu^2 & \frac{1}{2}\mu(1-\mu) \\ \\ \frac{1}{4}(1-\mu)^2 & \frac{1}{2}\mu(1-\mu) & \frac{1}{4}(1-\mu)^2 \end{array}$$

Passing a field with this operator is identical to

$$\bar{v} = v + \frac{1}{2}(1-\mu)\Delta^2 \left[\mu\left(v_{xx} + v_{yy}\right) + (1-\mu)\left(\overline{v}^{yy}_{xx} + \overline{v}^{xx}_{yy}\right)\right]$$

which shows the equivalence of this type of smoothing to diffusion. The plan of the finite difference estimate of the Laplacian is

$$\nabla^2 v \cong \mu\,(v_{xx} + v_{yy}) + (1-\mu)\left(\overline{v}^{yy}_{xx} + \overline{v}^{xx}_{yy}\right):$$

$$\begin{array}{ccc} \frac{1}{2}(1-\mu) & \mu & \frac{1}{2}(1-\mu) \\ \\ \mu & -2(1+\mu) & \mu \\ \\ \frac{1}{2}(1-\mu) & \mu & \frac{1}{2}(1-\mu) \end{array}$$

From (2) K is related to $\mu$ by

$$2\,K\,\Delta t = \frac{1}{2}(1-\mu)\,\Delta^2$$

What about so-called smoothing-unsmoothing? I will answer two questions here. (1) Is it related to diffusion? (2) Is there a way to do it when only 3 rows of the grid are available at a time?

The answer to the first question is no. This can be seen in the one-dimensional case. In one dimension, the plan of a smoothing, or unsmoothing operator is

$$\frac{1}{2}(1-\mu) \qquad \mu \qquad \frac{1}{2}(1-\mu) \tag{5a}$$

The plan of the corresponding unsmoothing operator is

$$-\frac{1}{2}(1-\mu) \qquad (2-\mu) \qquad -\frac{1}{2}(1-\mu) \tag{5b}$$

The plan of the two combined is

$$-\tfrac{1}{4}(1-\mu)^2 \qquad (1-\mu)^2 \qquad 1-\tfrac{3}{2}(1-\mu)^2 \qquad (1-\mu)^2 \qquad -\tfrac{1}{4}(1-\mu)^2$$

which is equivalent to

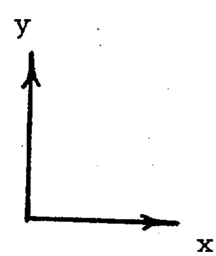
$$\bar{v} = v + (1-\mu)^2 \, \Delta^2 \, (v_{xx} - v_{2x2x}) \tag{5c}$$

The second term is clearly not a diffusion term.

The answer to the second question is "probably." One approach would be to change the sign of K in (2) every time step. This would amount to alternating smoothing and unsmoothing at odd and even time steps. You might get into trouble with this if the solutions tended to separate at odd and even steps, however. To overcome this, the sign of K could be given the following pattern in time

$$\ldots ++--++--++\ldots \tag{6}$$

To reduce the unsmoothing in a given time step, a two dimensional operator that smooths in one dimension and unsmooths in the other could be used in place of the finite-difference Laplacian. The alternation with time-step would then be on the dimension that is smoothed or unsmoothed. The smoother (5a) applied in the x-dimension and the unsmoother (5b) in the y-dimension has the plan




$$\begin{array}{ccc}
-\tfrac{1}{4}(1-\mu)^2 & -\tfrac{1}{2}\mu(1-\mu) & -\tfrac{1}{4}(1-\mu)^2 \\[2em]
\tfrac{1}{2}(1-\mu)(2-\mu) & 1-(1-\mu)^2 & \tfrac{1}{2}(1-\mu)(2-\mu) \\[2em]
-\tfrac{1}{4}(1-\mu)^2 & -\tfrac{1}{2}\mu(1-\mu) & -\tfrac{1}{4}(1-\mu)^2
\end{array}$$

which is equivalent to

$$\bar{v} = v + (1-\mu)\, L(v)$$

where the operator L has the plan

$$\begin{array}{ccc}
-\tfrac{1}{4}(1-\mu) & -\tfrac{1}{2}\mu & -\tfrac{1}{4}(1-\mu) \\[2em]
\tfrac{1}{2}(2-\mu) & -(1-\mu) & \tfrac{1}{2}(2-\mu) \\[2em]
-\tfrac{1}{4}(1-\mu) & -\tfrac{1}{2}\mu & -\tfrac{1}{4}(1-\mu)
\end{array}$$

With the 6L PE and LFM-I, satisfactory smoothing weights were experimentally determined for a smoother-unsmoother. If now the smoothing is done on only half of the time steps, and the unsmoothing on the other half, then the weights must be adjusted to achieve the same effect. The adjustments of the weights need not be determined experimentally, but can be calculated.

Consider the one-dimensional smoother-unsmoother (5c). When applied to

$$v = A\, e^{irx}$$

the result is

$$\overline{v} = v\,(1 - \nu^2\, \xi^2)$$

where

$$\nu = 1 - \mu$$

$$\xi = 1 - \cos r\,\Delta$$

Shuman (1957) called $\nu$ the smoothing index. Now if $\nu_1$ is the smoothing index used for a smoother-unsmoother done every time step, then to achieve the same effect, with $\nu_2$ in half of the time steps,

$$(1 - \nu_1^2\, \xi^2)^2 \cong 1 - \nu_2^2\, \xi^2$$

or

$$1 - 2\,\nu_1^2\, \xi^2 + \nu_1^4\, \xi^4 \cong 1 - \nu_2^2\, \xi^2$$

Thus, if $\nu_1\, \xi$ is small,

$$\nu_2 = \nu_1 \sqrt{2}$$

In the case of LFM-I, $\nu_1 = 0.2$. Thus, to achieve the same effect in LFM-II with smoothing and unsmoothing alternating,

$$\nu_2 = 0.2\,\sqrt{2} = 0.28284$$

The table below shows how well the two responses fit each other, where N is the number of time steps.

| Wave-length (grid intervals) | Response $\nu_1 = 0.2$ | | Response $\nu_2 = 0.2\sqrt{2}$ | |
|---|---|---|---|---|
| | N = 2 | N = 144 | N = 2 | N = 144 |
| 2 | .7056 | .0000 | .6800 | .0000 |
| 3 | .8281 | .0000 | .8200 | .0000 |
| 4 | .9216 | .0028 | .9200 | .0025 |
| 6 | .9801 | .2352 | .9800 | .2335 |
| 8 | .9931 | .6096 | .9931 | .6091 |
| 10 | .9971 | .8104 | .9971 | .8103 |
| 15 | .9994 | .9579 | .9994 | .9578 |
| 20 | .9998 | .9863 | .9998 | .9863 |
| 100 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

I have not dealt with the questions of stability nor loss of ordinary accuracy in alternation of smoothing and unsmoothing with the time step. Although such questions are valid ones, I will argue for now that with slight smoothing (and unsmoothing), and with the small time steps used, that an alternating system will behave virtually the same as the old system. An analysis is worth doing, however, and I intend to give it a try. Meanwhile, I have no proprietary rights to the problem.

If the alternation idea works, the same principle can be applied to develop more sophisticated filters by sequencing 9-point operators with the time step (Shuman, 1976).

References

Shuman, F. G., 1957: Numerical Methods in Weather Prediction: II. Smoothing and Filtering. Mon. Wea. Rev., 85, 357-361.

Shuman, F. G., 1976: How to Avoid Complex Smoothing Operators. Office Note 125, National Meteorological Center, 6 pp.